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NEW SOLUTIONS OF ANGULAR TEUKOLSKY EQUATION VIA TRANSFORMATION TO HEUNS EQUATION WITH THE APPLICATION OF RATIONAL POLYNOMIAL OF AT MOST DEGREE 2 TOGETHER WITH AN INTEGRAL OPERATOR

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ABSTRACT

The perturbation equation of masseless fields for Kerr-de Sitter geometry are written in form of seperable equations as in [19] called the Angular Teukolsky equation. The Angular Teukolsky equation is converted to General Heun's equation with singularities coinciding through some confluent process of one of five singularities. As in [17] and [18] rational polynomials of at most degree two are introduced.

AMS Subject classification: 33XX.

KEYWORDS: Heun Equation, Teukolsky Equation, Type-D Metrics, Polynomial Solutions

1. INTRODUCTION

Teukolsky equations are the consequences of perturbation equation fer Kerr- de Sitter geometry with the separability of angular and radial parts respec- tively. Carter [1] was the first to discover that the scalar wave function is Separable other consideration is the $\frac{1}{2}$ Spin electromagnetic field, gravita- tional perturbations and gravitino for the Kerr-deSitter class of geometry.

The Teukolsky equation is applicable in he study of black holes in gen-eral. The solutions of the equation are in most cases expressed as series solutions of some specialized functions. This approach has been carried out by so many researchers say Teukolsky (1973), Breuer et all (1977), Frackerelland Crossman (1977), Leahy and Unruh (1979), Chakrabarti (1984),Siedel(1989), Suzuki et all (1989) just to mention but few.AlthoughTeukolskyequationhasfivesingularpointsoneirregularwithfourregularpoints.Bysome confluent process, these singular points are reduced to four coinciding with the singular points of Heun' s equation.

The objective of this work is to obtain polynomial solutions for the de- rived Tuekolsky equation through its conversion to Heun' s equation through rational polynomials of degree at most 2. New solutions in terms of the ra- tional polynomials are obtained

The paper is organized as follows; The first section deals with the intro- duction of Teukolsky equation, the second section deals with the derivation of Teukolsky using the work of [19], the third section has to do with the derivation of Angular Teukolsky and its conversion to Heun' s equation and the fourth section has to do with Heun' s differential equation and its trans- formation to hypergeometric differential equation via rational polynomials of at most degree two.

2. THE TEUKOLSKYEQUATION [19]

Tekolsky equation was derived using the Kerr (-Newman)-de Sitter geome- tries

$$ds^2 = -p^2(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}) - \frac{\Delta_\theta \sin^2\theta}{(1+\alpha)^2 p^2}[adt - (r^2+a^2)d\varphi]^2 + \frac{\Delta_r}{(1+\alpha)^2\rho^2}(dt - a\sin^2\theta d\varphi)^2, \quad (1)$$

Where

$$\Delta_r = (r^2+a^2)(1-\frac{a}{ar^2}r^2) - 2Mr + Q^2 = -\frac{\alpha}{a^2}(r-r_+)(r-r_-)(r-r'_+)(r-r'_-),$$
$$\Delta_\theta = 1 + a\cos^2\theta, \alpha = \frac{\Lambda a^2}{3}, \bar{\rho} = r + ia\cos\theta \text{ and } \rho^2 = \bar{\rho}\bar{\rho}. \quad (2)$$

Here Λ is the cosmological constant, M is the mass of the black hole, rM its angular momentum and Q its charge. The electromagnetic field due to the charge of the black hole was given by

$$A_\mu dx^\mu = -\frac{Qr}{(1+\alpha)^2\rho^2}(dt - a\sin^2\theta d\varphi). \quad (3)$$

In particular, the following vectors were adopted as the null tetrad,

$$\iota^\mu = (\frac{(1+\alpha)(r^2+a^2)}{\Delta_r}, 1, 0, \frac{a(1+\alpha)}{\Delta_r}),$$
$$n^\mu = \frac{1}{2\rho^2}((1+\alpha)(r^2+a^2), -\Delta_r, 0, a(1+\alpha))$$
$$m^\mu = \frac{1}{\bar{p}\sqrt{2\Delta_\theta}}(ia(1+\alpha)\sin\theta, 0, \Delta_\theta, \frac{i(1+\alpha)}{\sin\theta})\bar{m}^\mu = m^{*\mu}. \quad (4)$$

It was assumed that the time and azimuthal dependence of the fields has the form $e^{-i(\omega t - m\phi)}$,the tetrad components of derivatives and the electro-magnetic field were

$$\iota^\mu = D_0,\ n^\mu\partial_\mu = \frac{\Delta_r}{2\bar{\rho}}D_0^\dagger,\ m^\mu\partial_\mu = \frac{\sqrt{\Delta_\theta}}{\sqrt{2}\bar{\rho}}L_0^\dagger,$$
$$m^\mu\partial_\mu = \frac{\sqrt{\Delta_\theta}}{\sqrt{2}p^*}L_0,\ \iota^\mu A_\mu = -\frac{Qr}{\Delta_r},\ n^\mu A_\mu = -\frac{Qr}{2\rho^2},$$
$$m^\mu A_\mu = \bar{m}^\mu A_\mu = 0, \quad (5)$$

Where

$$D_n = \partial_r - \frac{\imath(1+\alpha)K}{\Delta_r} + n\frac{\partial_r\Delta_r}{\Delta_r},\ D_n^\dagger = \partial_r + \frac{\imath(1+\alpha)K}{\Delta_r} + n\frac{\partial_r\Delta_r}{\Delta_r},$$

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$$L_n = \partial_\theta + \frac{\imath(1+\alpha)H}{\Delta_\theta} + n\frac{\partial_\theta(\sqrt{\Delta_\theta}\sin\theta)}{\sqrt{\Delta_\theta}\sin\theta},$$
$$L_n^\dagger = \partial_\theta - \frac{\imath(1+\alpha)H}{\Delta_\theta} + n\frac{\partial_\theta(\sqrt{\Delta_\theta}\sin\theta)}{\sqrt{\Delta_\theta}\sin\theta}, \tag{6}$$

with $K = \omega(r^2 + a^2) - am$ and $H = -a\omega\sin\theta + \frac{m}{\sin\theta}$.
Using the Newman-Penrose formalism it was showned that perturbation

equation in the Kerr-de sitter geometry are separable for massless spin 0, $\frac{1}{2}$,1,$\frac{3}{2}$ and 2 fields. Similarly in the Kerr-Newman-de sitter space those for spin 0,$\frac{1}{2}$ fields are also separable. The separated equations for fields with spin s and charge e were given by

$$[\sqrt{\Delta_\theta}L_{1-s}^\dagger\sqrt{\Delta_\theta}L_s$$
$$-2(1+\alpha)(2s-1)a\omega\cos\theta - 2\alpha(s-1)(2s-1)\cos^2\theta + \lambda]S_s(\theta) = 0$$
$$[\Delta_r D_1 D_s^\dagger + 2(1+\alpha)(2s-1)\imath\omega - \frac{2\alpha}{a^2}(s-1)(2s-1)$$
$$+\frac{-2(1+\alpha)eQKr + \imath seQr\partial_r\Delta_r + e^2Q^2r^2}{\Delta_r} - 2\imath seQ - \lambda]R_s(r) = 0. \tag{7}$$

3. TRANSFORMATION OF TEUKOLSKY EQUATION TOHEUN'S EQUATION [19]

It was shown in [19] that the Teukolsky equations after separation can be transformed to the Heun's equation by factoring out a single regular singu- larity.

2.1 Angular Teukolsky Equation

From (7), the angular Teukolsky equation after seperation was shown to be

$$\Big\{\frac{d}{dx}[(1+\alpha x^2)(1-x)\frac{d}{dx}] + \lambda - s(1-\alpha) + \frac{(1+\alpha)^2}{\alpha}\xi^2 - 2\alpha x^2$$
$$+\frac{1+\alpha}{1+\alpha x^2}\Big[s(\alpha m - (1+\alpha)\xi)x - \frac{(1+\alpha)^2}{\alpha}\xi^2 - 2m(1+\alpha)\xi + s^2\Big]$$
$$-\frac{(1+\alpha)^2m^2}{(1+\alpha x^2)(1-x^2)} - \frac{(1+\alpha)(s^2+2smx)}{1-x^2}\Big\}S(x) = 0, \tag{8}$$

Where $x = \cos\theta$ and $\xi = a\omega$. This equation has five regular singularitie as ±1, ± I and ∞. It was also noted that the angular equation has no Independence on M ad Q By choosing the variable z such as

$$z = \frac{1-\frac{\imath}{\sqrt{a}}}{2}\frac{x+1}{x-\frac{\imath}{\sqrt{a}}},$$

Then (8) takes the following form,

$$\Big\{\frac{d^2}{dz^2}+\Big[\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-z_s}-\frac{2}{z-z_\infty}\Big]\frac{d}{dz}$$

$$-\Big(\frac{m-s}{2}\Big)^2\frac{1}{z^2}-\Big(\frac{m+s}{2}\Big)^2\frac{1}{(z-1)^2}+\Big(\frac{1+\alpha}{\sqrt{\alpha}}\xi-\frac{\sqrt{\alpha}m+\imath s}{2}\Big)^2\frac{1}{(z-z_s)^2}+$$

$$\frac{2}{(z-z_\infty)^2}+\Big[-\frac{m^2}{2}\Big(1+\frac{4\alpha}{(1+\imath\sqrt{\alpha})^2}\Big)+\frac{s^2}{2}\Big(\frac{1-\imath\sqrt{a}}{1+\imath\sqrt{\alpha}}\Big)^2+\frac{2\imath ms\sqrt{\alpha}}{1+\imath\sqrt{\alpha}}$$

$$+\frac{\lambda-s(1-\alpha)-2\alpha+2(1+\alpha)(m+s)\xi}{(1+\imath\sqrt{\alpha})^2}\Big]\frac{1}{z}$$

$$+\Big[\frac{m^2}{2}\Big(1+\frac{4\alpha}{(1-\imath\sqrt{\alpha})^2}\Big)-\frac{s^2}{2}\Big(\frac{1+\imath\sqrt{a}}{1-\imath\sqrt{\alpha}}\Big)^2-\frac{2\imath ms\sqrt{\alpha}}{1-\imath\sqrt{\alpha}}$$

$$-\frac{\lambda-s(1-\alpha)-2\alpha+2(1+\alpha)(m-s)\xi}{(1+\imath\sqrt{\alpha})^2}\Big]\frac{1}{z-1}$$

$$+\Big[-m^2\frac{2\imath\sqrt{\alpha}m}{(1-\alpha)^2}-s^2\frac{\sqrt{\alpha}(1-\alpha)}{(1+\alpha)^2}-ms\frac{\alpha}{1+\alpha}+\frac{\imath\sqrt{\alpha}(\lambda-s(1-\alpha)+2)}{(1+\alpha)^2}$$

$$+\frac{(2\imath\sqrt{\alpha}m+(\alpha-1)s)\xi}{1+\alpha}\Big]\frac{4}{z-z_s}-\frac{8\imath\sqrt{\alpha}}{1+\alpha}\frac{1}{z-z_\infty}\Big\}S(z)=0, \tag{9}$$

where $A_1=\frac{|m-s|}{2}$, $A_2=\frac{|m+s|}{2}$, and $A_3=\pm\frac{\imath}{2}\Big(\frac{1+\alpha}{\sqrt{\alpha}\xi}-\sqrt{\alpha}m-\imath s\Big)$. Now $f(z)$ satisfies the equation

$$\Big\{\frac{d}{dz^2}+\Big[\frac{2A_1+1}{z}+\frac{2A_2+1}{z-1}+\frac{2A_3+1}{z-z_s}\Big]\frac{d}{dz}+\frac{\rho_\pm z+u}{z(z-1)(z-z_s)}\Big\}f(z)=0, \tag{10}$$

Where

$$\rho_\pm=A_1+A_2+A_3\pm A_3^*+1$$

,

$$u=\frac{-\imath}{4\sqrt{\alpha}}\Big\{\lambda-s(1-\alpha)-2\imath\sqrt{\alpha}+2(1+\alpha)(m+s)\xi-(1+\imath\sqrt{\alpha})^2(2A_1A_2+A_1+A_2)$$

$$-4\imath\sqrt{\alpha}(2A_1A_3+A_1+A_3)-\frac{m^2}{2}[4\alpha+1+(\imath\sqrt{\alpha})^2]$$

$$+\frac{s^2}{2}(1-\imath\sqrt{\alpha})^2+2\imath ms\sqrt{\alpha}(1+\imath\sqrt{\alpha})\Big\}.$$

Equation (10) is called the Heun's equation which has four regular singu- larities. The f(z) is determined by requiring non-singular behaviors at z = 0 and 1. We can take either one of signs of $A3$ to find the solution S (z) in terms of solution of Heun's differential equation.

4. HEUN'SEQUATIONTOHYPERGEOMETRICVIARATIO-NAL POLYNOMIALTRANSFORMATIONS

In this section, we transform the Heun's equation derived above to hyperge- ometric differential equation with three singularities and back again to the Heun's solutions with polynomial terms.

The hyper geometric equation has three regular singular points. Heun's equation has four regular points. The problem of conversion from Heun's equation to hyper geometric equation has been treated in the works of K. Kuiken [17]. The purpose of this work is to derive some forms solution to the Heun's equation via some rational transformation as stated earlier.

The steps taken shall be conversion of Heun's function to the hyper geomet- ric function then taken the derivatives, and through a push and pull back process we arrive back to a new Heun's function different from the original Heun'sfunctions.

Every homogenous linear second order differential equation with four regular singularities can be transformed into (10) with the assumption that $2A_1 + 1 = \gamma$, $2A_2 + 1 = \delta$, $2A_3 + 1 = s$, $\rho_\pm = \alpha\beta$ and $u = q$, $z = t$ and $z_s = d$ as defined above, and read as

$$\frac{d^2u}{dt^2} + (\frac{\gamma}{t} + \frac{\delta}{t-1} + \frac{\epsilon}{t-d})\frac{du}{dt} + \frac{\alpha\beta t - q}{t(t-1)(t-d)}u = 0, \tag{11}$$

Where $\{\alpha, \beta, \gamma, \delta, s, d, q\}(d\ f= 0, 1)$ are parameters, generally complex and ar- bitrary, linked by FUSCHAIN constraint $\alpha+\beta+1 = \gamma+\delta+s$. This equation Has four regular singular points at $\{0, 1, a, \infty\}$, with the exponents of these singular being respectively, $\{0, 1, -\gamma\}$, $\{0, 1, -\delta\}$, $\{0, 1, -s\}$ and $\{\alpha, \beta\}$. The equation (11) is called Heun's equation.

The Hyper geometric equation

$$z(1-z)\frac{d^2u}{dt^2} + [c - (a+b+1)z]\frac{dy}{dz} - aby = 0, \tag{12}$$

Has three regular singular points. in the above (11), it has been shown that these two equation above can be transformed to one another via six rational Polynomial $z = R(t)$, where $R(t) = t^2$, $1-t^2$, $(t-1)^2$, $2t-t^2(2t-1)^2$, $4t(1-t)$. The following parameter relations were deduced.

For the polynomial $R(t) = t^2$,

- $\alpha+\beta=2(a+b), \alpha\beta=4ab, \gamma=-1+2c, \delta=1+a+b-c, \delta s=\delta, q=0$ and $d=-2$.

For the polynomial $R(t) = 1 - t^2$

- $\alpha + \beta = 2(a + b)$, $\alpha\beta = 4ab$, $\gamma = -1 - 2c + 2a + 2b$, $\delta = c$, $s = \delta$, $q = 0$ and $d = 12$.

For the polynomial $R(t) = 2t - t^2$

- $\alpha+\beta = 2(a+b)$, $\alpha\beta = 4ab$, $\gamma = c$, $\delta = 1-2c+2a+2b$, $s = \delta = c$, $q = 4ab$ and $d = 2$.

For the polynomial $R(t) = (2t - 1)^2$

- $\alpha+\beta = 2(a+b)$, $\alpha\beta = 4ab$, $\gamma = -1+a+b-c$, $\delta = \gamma$, $\delta = s = -1$, $q = 4ab$ and $d = \underline{1}$.

For the polynomial $R(t) = 4t(1 - t)^2$

- $\alpha + \beta = 2(a + b)$, $\alpha\beta = 4ab$, $\gamma = c$, $\delta = \gamma$, $\delta = 1 - 2c + 2a + 2b$, $q = 2ab$ and $d = \underline{1.}$

Assuming $H(d, q, \alpha, \beta, \gamma, \delta, s; t) = S_i(t)$; $i = 1 \ldots 14$ are solutions of the Angular Teukols kyinterms of Heun's with polynomial factor and $_2F_1(a,b;c;z= R(t))$ are representative forms of the solutions of (11) and (12) respective, together with parameters above relations can be established between these two forms via the polynomials data given above. We provide an answer to this in this paper. Indeed, we provide that the derivative of the solution of Heun's can be expressed in terms of another Heun's solution giving rise to new solutions of Teukolsky Angul are quation.

INTEGRAL SOLUTIONS TO HEUN'S DIFFERENTIAL

2. Main Results: Integral Solutions

In this section we shall apply the relations above in deriving the integral form of solutions via these polynomial transformations. Let $I = \int_C$ be an integral operator defined over a compact interval C. Since $(a)_{n-1} = \frac{(a-1)_n}{a-1}$, we have

$$I_2F_1(a, b; c; z = R(t)) = \frac{R^*(t)(c-1)}{(a-1)(b-1)} 2F_1(a-1, b-1; c-1; z = R(t)),$$

Where $R^*(t)$ is a polynomial factor derived from the integrand and through a push and pull-back processes we have the following possible solutions of the Angular Teukolsky equstion;

1. For polynomial $R(t) = t^2$:
   (a) Using $c = (\gamma+1)/2$, we obtain

$$\begin{aligned} & IH(-1, 0; \alpha, \beta, \gamma, \delta, \epsilon; t) \\ &= \tfrac{2(\gamma-1)t^3}{3(\alpha-2)(\beta-2)} 2F_1(\tfrac{\beta-2}{2}, \tfrac{\alpha-2}{2}; \tfrac{\gamma-1}{2}; R(t) = t^2)|_C \\ &= \tfrac{2(\gamma-1)t^3}{3(\alpha-2)(\beta-2)} H(-1, 0; \alpha-2, \beta-2, \gamma-2, \\ &\qquad \frac{\alpha+\beta-\gamma-1}{2}, \frac{\alpha+\beta-\gamma-1}{2}; t)|_C = S_1. \end{aligned} \tag{13}$$

(b) Using $c = 1 - \delta + a + b$, we get

$$IH(-1, 0; \alpha, \beta, \gamma, \delta; \epsilon, t)$$
$$= \frac{4(\alpha+\beta-2\delta)t^3}{3(\alpha-2)(\beta-2)} 2F_1(\frac{\beta-2}{2}, \frac{\alpha-2}{2}; \alpha+\beta-2\delta; R(t) = t^2)|_C$$
$$= \frac{4(\alpha+\beta-2\delta)t^3}{3(\alpha-2)(\beta-2)} \times H(-1, 0; \alpha-2, \beta-2, 2(\alpha+\beta-2\delta)-1, \frac{4\delta-(\alpha+\beta)-2}{2}, \frac{4\delta-(\alpha+\beta)-2}{2}; t)|_C = S_2. \tag{14}$$

2. For polynomial $R(t) = 1 - t^2$:

(a) Using $c = \delta$, we have

$$IH(-1, 0; \alpha, \beta, \gamma, \delta, \epsilon; t)$$
$$= \frac{4(\delta-1)(3t-t^3)}{3(\alpha-2)(\beta-2)} 2F_1(\frac{\beta-2}{2}, \frac{\alpha-2}{2}; \delta-1; R(t) = 1-t^2)|_C$$
$$= \frac{4(\delta-1)(3t-t^3)}{3(\alpha-2)(\beta-2)} \times H(-1, 0; \alpha-2, \beta-2, \alpha+\beta-2\delta-1, \delta-1, \delta-1; t)|_C = S_3. \tag{15}$$

(b) Using $c = \epsilon$, we have

$$`\ IH(-1, 0; \alpha, \beta, \gamma, \delta, \epsilon; t)$$
$$= \frac{4(\epsilon-1)(3t-t^3)}{3(\alpha-2)(\beta-2)} 2F_1(\frac{\beta-2}{2}, \frac{\alpha-2}{2}; \epsilon-1; R(t) = 1-t^2)|_C$$
$$= \frac{4(\epsilon-1)(3t-t^3)}{3(\alpha-2)(\beta-2)} \times H(-1, 0; \alpha-2, \beta-2, \alpha+\beta-2\epsilon-1, \epsilon-1, \epsilon-1; t)|_C = S_4. \tag{16}$$

(c) Using $c = (1 - \gamma + 2a + 2b)/2$, we arrive at

$$IH(-1, 0; \alpha, \beta, \gamma, \delta, \epsilon; t)$$
$$= \frac{2(\alpha+\beta-\gamma-1)(3t-t^3)}{3(\alpha-2)(\beta-2)} 2F_1(\frac{\beta-2}{2}, \frac{\alpha-2}{2}; \frac{\alpha+\beta-\gamma-1}{2}; R(t) = 1-t^2)|_C$$
$$= \frac{2(\alpha+\beta-\gamma-1)(3t-t^3)}{3(\alpha-2)(\beta-2)} \times H(-1, 0; \alpha-2, \beta-2, \gamma-2, \frac{\alpha+\beta-\gamma-1}{2} \frac{\alpha+\beta-\gamma-1}{2}; t)|_C = S_5. \tag{17}$$

3. For polynomial $R(t) = 2t - t^2$:

   (a) Using $c = (\delta + 1)/2$, we obtain

$$IH(2, \alpha\beta; \alpha, \beta, \gamma, \delta, \epsilon; t)$$
$$= \tfrac{2(\delta-1)t^2(3t-t^2)}{3(\alpha-2)(\beta-2)} 2F_1(\tfrac{\beta-2}{2}, \tfrac{\alpha-2}{2}; \tfrac{\delta-1}{2}; R(t) = 2t - t^2)|_C$$
$$= \tfrac{2(\delta-1)t^2(3t-t^3)}{3(\alpha-2)(\beta-2)} H(2, (\alpha-2)(\beta-2); \alpha-2, \beta-2, \tfrac{\alpha+\beta-\delta-1}{2}, \delta-2,$$
$$\frac{\alpha+\beta-\delta-1}{2}; t)|_C = S_6. \tag{18}$$

   (b) Using $c = 1 + a + b - \gamma$, we have

$$IH(2, \alpha\beta; \beta, \alpha, \gamma, \delta, \epsilon; t)$$
$$= \tfrac{2(\alpha+\beta-2\gamma)t^1(3-t^2)}{3(\alpha-2)(\beta-2)} 2F_1(\tfrac{\beta-2}{2}, \tfrac{\alpha-2}{2}; \gamma+1; R(t) = 2t - t^2)|_C$$
$$= \tfrac{2(\alpha+\beta-2\gamma)t^2(3-t^2)}{3(\alpha-2)(\beta-2)} H(2, (\alpha-2), (\beta-2); \alpha-2, \beta-2, \tfrac{\gamma-2}{2},$$
$$\alpha+\beta-\gamma-1, \frac{\gamma-2}{2}; t)|_C = S_7. \tag{19}$$

4. For polynomial $R(t) = (t-1)^2$:

   (a) Using $c = (1 - \delta + 2a + 2b)/2$, we have

$$IH(2, \alpha\beta, \alpha, \beta, \gamma, \delta, \epsilon; t)$$
$$= \tfrac{2(\alpha+\beta-\delta-1)(t-1)^3}{3(\alpha-2)(\beta-2)} 2F_1(\tfrac{\beta-2}{2}, \tfrac{\alpha-2}{2}; \tfrac{\alpha+\beta-\delta-1}{2}; R(t) = (t-1)^2)|_C$$
$$= \tfrac{2(\alpha+\beta-\delta-1)(t-1)^3}{3(\alpha-2)(\beta-2)} \times H(2, (\alpha-2)(\beta-2); \alpha-2, \beta-2, \tfrac{\alpha+\beta-\delta-1}{2}, \tfrac{\alpha+\beta-\delta-1}{2},$$
$$\frac{\alpha+\beta-\delta-1}{2}; t)|_C = S_8. \tag{20}$$

   (b) Using $c = \gamma$, we have

$$IH(2, \alpha\beta; \alpha, \beta, \gamma, \delta, \epsilon; t)$$
$$= \tfrac{2(\gamma-1)(t-1)^3}{3(\alpha-2)(\beta-2)} 2F_1(\tfrac{\beta-2}{2}, \tfrac{\alpha-2}{2}; \gamma-1; R(t) = (t-1)^2)|_C$$
$$= \tfrac{2(\gamma-1)}{(\alpha-2)(\beta-2)} \times H(2, (\alpha-2)(\beta-2); \alpha-2, \beta-2, \gamma-1,$$
$$\alpha+\beta-2\gamma-1, \alpha+\beta-2\gamma-1; t)|_C = S_9. \tag{21}$$

(c) By changing γ toϵ in above, similar relation can be obtained.

5. For polynomial $R(t) = (2t-1)^2$:

   (a) Using $c = (\epsilon+1)/2 = (\delta+1)/2$, we have

$$IH(1/2, \alpha\beta/2; \alpha, \beta, \gamma, \delta, \epsilon; t)$$
$$= \tfrac{2(\epsilon-1)(2t-1)^3}{6(\alpha-2)(\beta-2)} 2F_1(\tfrac{\beta-2}{2}, \tfrac{\alpha-2}{2}; \tfrac{\epsilon-1}{2}; R(t) = (2t-1)^2)|_C$$
$$= \tfrac{2(\epsilon-1)(2t-1)^3}{6(\alpha-2)(\beta-2)} \times H(1/2, \tfrac{(\alpha-2)(\beta-2)}{2}; \alpha-2, \beta-2, \tfrac{\alpha+\beta-\epsilon-5}{2},$$
$$\frac{\alpha+\beta-\epsilon-5}{2}, \epsilon-2; t)|_C = S_{10}. \quad (22)$$

By changing ϵ to δ a similar expression can be obtained.

   (b) Using $c = -1 + a + b - \gamma$, we obtain

$$IH(1/2, \alpha\beta/2; \alpha, \beta, \gamma, \delta, \epsilon; t)$$
$$= \tfrac{2(\alpha+\beta-2(\gamma+2))(2t-1)^3}{6(\alpha-2)(\beta-2)} 2F_1(\tfrac{\beta-2}{2}, \tfrac{\alpha-2}{2}; \tfrac{\alpha+\beta-2\gamma-4}{2}; R(t) = (2t-1)^2)|_C$$
$$= \tfrac{2(\alpha+\beta-2(\gamma+2))(2t-1)^3}{6(\alpha-2)(\beta-2)} \times H(1/2, \tfrac{(\alpha+2)(\beta+2)}{2}; \alpha-2,$$
$$\beta-2, \gamma-1, \alpha+\beta-2\gamma-5; t)|_C = S_{11}. \quad (23)$$

6. For polynomial $R(t) = 4t(1-t)$:

   (a) Using $c = \gamma$, we have

$$IH(1/2, \alpha\beta/2; \beta, \alpha, \gamma, \delta, \epsilon, ; t)$$
$$= \tfrac{4(\gamma-1)2t^2(3-2t)}{3(\alpha-2)(\beta-2)} 2F_1(\tfrac{\beta-2}{2}, \tfrac{\alpha-2}{2}; \gamma-1; R(t) = 4t(1-t))|_C$$
$$= \tfrac{4(\gamma-1)2t^2(3-2t)}{3(\alpha-2)(\beta-2)} \times H(1/2, \tfrac{(\alpha-2)(\beta-2)}{2}; \alpha-2, \beta-2,$$
$$\gamma-1, \gamma-1, \alpha+\beta-2\gamma-1; t)|_C = S_{12}. \quad (24)$$

   (b) $IH(1/2, \alpha\beta/2; \beta, \alpha, \gamma, \delta, \epsilon; t)$

$$= \tfrac{4(\delta-1)2t^2(3-2t)}{3(\alpha-2)(\beta-2)} 2F_1(\tfrac{\beta-2}{2}, \tfrac{\alpha-2}{2}; \delta-1; R(t) = 4t(1-t))|_C$$
$$= \tfrac{4(\delta-1)2t^2(3-2t)}{3(\alpha-2)(\beta-2)} \times H(1/2, \tfrac{(\alpha-2)(\beta-2)}{2}; \alpha-2, \beta-2,$$
$$\delta-1, \delta-1, \alpha+\beta-2\delta-1; t)|_C = S_{13}. \quad (25)$$

(c) Using $c = (1 - \epsilon + 2a + 2b)/2$,

$$IH(1/2, \alpha\beta/2; \alpha, \beta, \gamma, \delta, \epsilon, ; t)$$
$$= \frac{2(\alpha+\beta-\epsilon-1)2t^2(3-2t)}{3(\alpha-2)(\beta-2)} 2F_1(\frac{\beta-2}{2}, \frac{\alpha-2}{2}; \frac{\alpha+\beta-\epsilon-1}{2}; R(t) = 4t(1-t))|_C$$
$$= \frac{2(\alpha+\beta-\epsilon-1)2t^2(3-2t)}{3(\alpha-2)(\beta-2)} \times H(1/2, \frac{(\beta-2)(\alpha-2)}{2}; \alpha-2, \beta-2,$$
$$\frac{\alpha+\beta-\epsilon-1}{2}, \frac{\alpha+\beta-\epsilon-1}{2}, \epsilon-2; t)|_C = S_{14}. \tag{26}$$

3. CONCLUDING REMARKS AND SUGGESTIONS

In this paper, we have shown that the solutions of the derived Angular Teukolsky equation transformed to Heun's equation could be obtained in form of Heun's functions via polynomials of at most degree two Trans forma- tions with the application of integral operator *I*. The Heun's equation was initially compare with the hyper geometric differential equation with three singularities via the giving polynomials. Another result could be obtained if we apply the differential operator instead of the integral operator. Poly- nomials of higher degrees are being considered.

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